Simple computer model of a fractal river network with fractal individual watercourses

1993 J. Phys. A: Math. Gen. 26 L623

(http://iopscience.iop.org/0305-4470/26/15/002)

J. Phys. A: Math. Gen. 26 (1993) L623–L627. Printed in the UK

LETTER TO THE EDITOR

Simple computer model of a fractal river network with fractal individual watercourses

V B Sapozhnikov and V I Nikora

Institute of Geophysics and Geology, Academy of Sciences of RM, Academiei str 3, 277028 Kishinev, Republic of Moldova

Received 9 November 1992, in final form 22 February 1993

Abstract. A random walk computer model of river network is proposed. It is shown that the length and the width of both network and of individual streams that constitute it exhibit scaling $\bar{\sigma}_\parallel \sim \mathscr{L}^{\nu_\parallel}$ and $\bar{\sigma}_\perp \sim \mathscr{L}^{\nu_\perp}$ ($\bar{\sigma}_\parallel$ and $\bar{\sigma}_\perp$ are the longitudinal and lateral sizes of the object, $\mathscr{L}$ is the overall length of the object). The simulated individual streams display self-similar behaviour at small $\mathscr{L}$ ($\nu_\parallel = \nu_\perp = 0.80 \pm 0.03$) and self-affine behaviour at large $\mathscr{L}$ ($\nu_\parallel = 0.99 \pm 0.03$, $\nu_\perp = 0.50 \pm 0.03$). Similar behaviour is observed for simulated river networks too: $\nu_\parallel = \nu_\perp = 0.66 \pm 0.03$ correspond to these in the self-similarity region, while in the self-affinity region $\nu_\parallel = 0.74 \pm 0.03$ and $\nu_\perp = 0.43 \pm 0.03$. Proceeding from the self-affinity of individual rivers and river networks Hack's empirical law $L \sim F^\beta$ has been substantiated (L is the length of the main river, F the catchment area), where $\beta = 1/(1+H)$, $H = \nu_\perp/\nu_\parallel$, Hurst's exponent for river networks. The scaling for the water mass distribution over the river network in the self-affine region is also revealed: $\bar{\sigma}_{m\parallel} \sim \mathscr{L}^{\nu_{m\parallel}}$, $\bar{\sigma}_{m\perp} \sim \mathscr{L}^{\nu_{m\perp}}$, $\nu_{m\parallel} = 0.72 \pm 0.03$, $\nu_{m\perp} = 0.38 \pm 0.03$. It is shown that in this region the water mass M depends upon the network total length and upon the catchment area as a power law: $M \sim \mathscr{L}^{1.67} \sim F^{1.43}$.

Computer models of fractal river networks have been previously studied in [1–4]. The simulated river networks obtained by the authors of these works were, in the majority of cases, both separately and in their aggregate, compact sets (they covered the area densely). Moreover, the individual river networks for several models proved to be self-affine fractal objects, the geometry of which was described by the relationships

$$l \sim \mathscr{L}^{\nu_\parallel} \tag{1}$$

$$b \sim \mathscr{L}^{\nu_\perp} \tag{2}$$

where l and b are the characteristic longitudinal and lateral sizes of the network, and $\mathscr{L}$ the overall length of the river network. In all cases the scaling indices $\nu_\parallel$ and $\nu_\perp$ proved to be equal to $\frac{2}{3}$ and $\frac{1}{3}$, respectively, which provided grounds for the authors of [3] to indicate the discovery of some universality class—'Scheidegger's river networks'. Besides this, the model of self-similar DLA with fractal dimension less than 2.0 was used as a river network model in [1]. In the present article we have considered the case of simulated river networks with fractal individual streams which do not fall under the described universality class.

The process of natural river network formation is very complicated and at present is not studied completely. It seems to us that its adequate simulation based on the fundamental laws of physics is hardly possible at this time. Therefore we tried to build the simplest model, which would, nevertheless, reproduce the main geometrical features of natural rivers. We proceeded from the fact that the stochastic character of the natural river is caused not only by irregularities of the land but also by the stochastic nature of the process of the river network formation itself. The river not only follows the land, but also forms it. Therefore in our model we did not consider the land irregularities as quenched. We tried to take into account two factors governing the river network formation: land irregularities and inertia of the flow.

When simulating the river network we used the random walk method. First, a certain number of particles (river sources) was randomly placed on a 300×300 square lattice. Then the simultaneous wandering of particles over the lattice was started, the trajectories of which were regarded as rivers. Each move, the particle jumped to the place of one of the neighbouring sites located on the sides and diagonals of the square cell. The relative probabilities of such jumps were determined as the product of two probabilities. The first simulated the relief of the slope on which the 'rivers' flowed. In the direction of the general land slope this probability was maximal. The second probability determined the turning of the trajectory with respect to the direction of the previous jump. It was maximal in this direction. The turning of the trajectory at each step by more than 90° was forbidden (since 'instantaneous' turning of a natural river through a large angle is unlikely). Nevertheless, turn of the simulated river through any angle may occur after several steps. The direction of each next jump was chosen randomly using the product of the above described probabilities. The first one characterizes, to some extent, the role of gravity, and the second one the role of inertial force.

When the walking particle crossed its own trajectory the formed loop was erased (to some extent this corresponds to the disappearance of river meanders after channel autopiracy). Unlike the models of other authors, in our model the particle was allowed to walk (following the local slope) in all directions. In the case of two-trajectory-confluence only one particle continued walking. The simulation process ended when all particles reached the lower edge of the lattice (ocean shore).

As a result, an aggregate of river networks of various sizes was formed within the simulated slope, which also included non-ramified rivers. One realization is given as an example in figure 1. We have investigated the river networks at various source

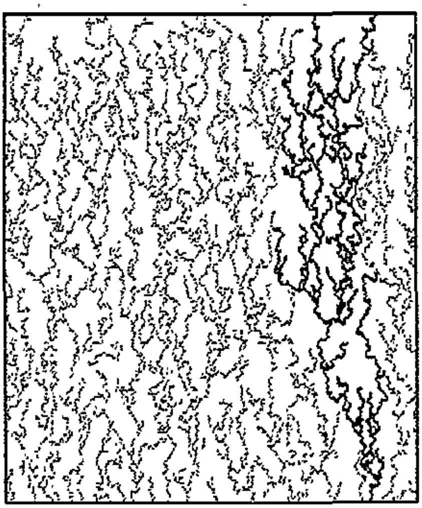

Figure 1. Example of simulated river netowrks. One of the individual river networks is highlighted.

densities and at various gradients of the simulated slope. The identification and analysis of both individual streams and individual river network fractal properties have been carried out according to one and the same technique using relationships

$$\bar{\sigma}_{\parallel} \sim \mathscr{L}^{\nu_{\parallel}} \tag{3}$$

$$\bar{\sigma}_{\perp} \sim \mathscr{L}^{\nu_{\perp}} \tag{4}$$

where $\bar{\sigma}_{\parallel}$ and $\bar{\sigma}_{\perp}$ are mean-square deviations of points belonging to an individual stream (or to an individual river network) with overall length $\mathscr{L}$ averaged over the streams or over the networks, respectively.

Analysis of graphs $\bar{\sigma}_{\parallel}(\mathscr{L})$ and $\bar{\sigma}_{\perp}(\mathscr{L})$ shows that at small $\mathscr{L}$ the simulated individual streams are characterized by self-similarity, and at large $\mathscr{L}$, by self-affinity. The scale of $\mathscr{L}_1$, which divides these two regions, decreases with increase of the overall slope. The characteristic graphs of functions $\bar{\sigma}_{\parallel}(\mathscr{L})$ and $\bar{\sigma}_{\perp}(\mathscr{L})$ for simulated individual streams are shown in figure 2(a). It has been established that in the region of self-similarity $\nu_{\parallel} = \nu_{\perp} = 1/D = 0.80 \pm 0.03$, i.e. the fractal dimension D is close to 1.25. For the self-affinity region the scaling exponents $\nu_{\parallel}$ and $\nu_{\perp}$ are equal to 0.99 ± 0.03 and 0.50 ± 0.03, respectively. The estimations obtained may be interpreted as follows. The excess of values $\nu_{\parallel}$ and $\nu_{\perp}$ in the self-similarity region (≈ 0.8) over the corresponding

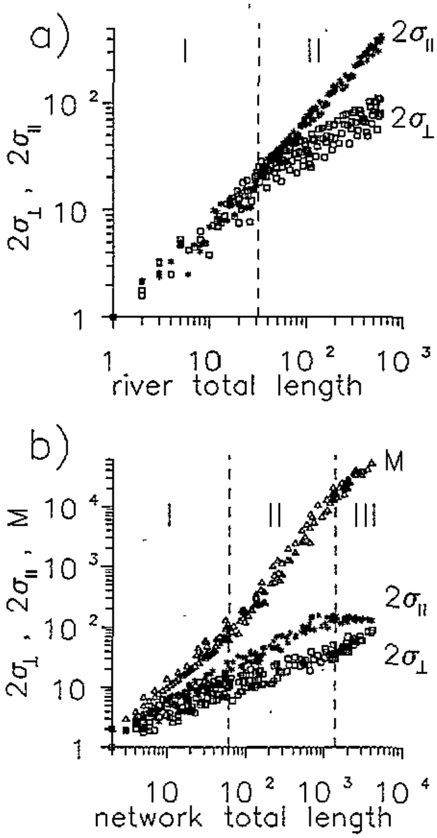

Figure 2. Characteristic graphs of functions $\sigma_{\parallel}(\mathscr{L})$, $\sigma_{\perp}(\mathscr{L})$ and $M(\mathscr{L})$ for individual rivers (a) and individual river networks (b). I is the self-similarity region, II the self-affinity region, III the edge effect region (the graphs show the points from to 10 simulations).

values for the Brownian motion (0.5) indicates that repulsion of the particle from its own trajectory takes place (this occurring by way of loop erasing).

The existence of a self-affinity region is explained by the following reasoning. The walking of the simulated stream trajectory is conditioned by two components—deterministic (overall gradient of the simulated slope) and random (local gradient connected with local roughness of the relief). On larger scales the influence of the deterministic component predominates, therefore, and $\nu_{\parallel}$ tends to 1.0. Meanwhile, the trajectory of the simulated stream on these scales ceases to experience repulsion from its distant parts and the $\nu_{\perp}$ exponent approaches 0.5 as in the case of Brownian motion. The values $\nu_{\parallel}$ and $\nu_{\perp}$ obtained for simulated individual streams for both the self-similarity and self-affinity regions are in good agreement with our data concerning the Dniester and Pruth rivers (self-similarity region: $\nu_{\parallel}=\nu_{\perp}=1/D=0.85\pm0.04$; self-affinity region: $\nu_{\parallel}=0.97\pm0.04$, $\nu_{\perp}=0.53\pm0.12$).

The analysis of graphs $\bar{\sigma}_{\parallel}(\mathscr{L})$ and $\bar{\sigma}_{\perp}(\mathscr{L})$ for simulated river networks also permitted the identification of two scaling regions—the self-similarity region at small $\mathscr{L}$ and the self-affinity region at large $\mathscr{L}$. The scale $\mathscr{L}_2$, which divides these regions, decreases with increase of the general gradient of the simulated slope. The characteristic graphs of functions $\bar{\sigma}_{\parallel}(\mathscr{L})$ and $\bar{\sigma}_{\perp}(\mathscr{L})$ for individual simulated river networks are shown in figure 2(*b*). For the self-similarity region $\nu_{\parallel}=\nu_{\perp}=0.66\pm0.03$. The $\nu_{\parallel}$ and $\nu_{\perp}$ exponents for the self-affinity region are equal to 0.74 ± 0.03 and 0.43 ± 0.03, respectively.

On a qualitative level the above-given estimations can be explained as was done when dealing with individual streams. For comparison with natural data we give values $\nu_{\parallel}=0.61$ and $\nu_{\perp}=0.44$ obtained for the Alto Liri basin (the data for calculating $\nu_{\parallel}$ and $\nu_{\perp}$ were taken from [5]). Thus, self-affinity is characteristic for both the simulated and natural river networks. In both cases the Hurst's exponent values [6] $H=\nu_{\perp}/\nu_{\parallel}$ appeared to be significantly less then 1.0 (0.58 and 0.72), while the lacunary dimension [7] $D_G=2/(\nu_{\parallel}+\nu_{\perp})$ was less than 2.0 (1.71 and 1.90).

The self-affinity of individual streams and river networks allows provision of the following substantiation to Hack's empirical law [8] $L\sim F^{\beta}$, where L is the length of the main river, F the catchment area, and $\beta\approx0.6$. Using the above-established relationships we can write

$$L\sim l^{1/\nu_{s\parallel}}\sim F^{\nu_{n\parallel}/\nu_{s\parallel}(n_{n\parallel}+\nu_{n\perp})} \tag{5}$$

so that

$$\beta=\nu_{n\parallel}/\nu_{s\parallel}(\nu_{n\parallel}+\nu_{n\perp})=1/\nu_{s\parallel}(1+H_n). \tag{6}$$

In the given relationship the '*s*' and '*n*' indices indicate that the corresponding exponents belong to individual streams ('*s*') or to river networks ('*n*'), $F\sim\bar{\sigma}_{n\parallel}\bar{\sigma}_{n\perp}$, $l\sim\bar{\sigma}_{n\parallel}$. As has been shown above, $\nu_{s\parallel}\sim1.0$ so we may write $\beta=1/(1+H_n)$. For the investigated simulated river networks we have $H_n=0.58$, from which $\beta=0.63$, in good agreement with the well known empirical estimations [8].

The fractal structure of the river network must also condition the scaling properties of hydrological characteristics, the geometrical carrier of which is the river network. Here we shall consider the scaling properties of mass M determined as the overall quantity of water in the river network:

$$M=\sum_{i=1}^{\mathscr{L}}\rho_i.$$

The linear density of mass ρ was taken as follows. For each point of tributaryless rivers (from source to the first influx of the tributary) $\rho=1$. With the inflow of each tributary the ρ value on the lower section of the river increased by the ρ value of the tributary. Within the tributaryless sections of the river the density was taken as constant. From the hydrological point of view such a scheme of ρ distribution within the river network is reasonably well substantiated [9] and thus it can be used to calculate M.

As a result of proper treatment of data in respect of simulated river networks for the self-affinity region relationships $M \sim \mathscr{L}^{1.67\pm0.05}$ (figure 2(b)), $\bar{\sigma}_{m\parallel} \sim \mathscr{L}^{0.72\pm0.03}$, $\bar{\sigma}_{m\perp} \sim \mathscr{L}^{0.38\pm0.03}$ were established. Here $\bar{\sigma}_{m\parallel}$ and $\bar{\sigma}_{m\perp}$ are the mean-square deviations of points belonging to the river network taken with weight ρ. The overscribed bar indicates averaging over the networks having overall length $\mathscr{L}$. In the self-similarity region we were unable to obtain similar relationships because of the small sizes of river networks.

These results allow overall water mass in the river network M and its catchment area F for the self-affinity region: $M \sim F^{1.43}$. The results shown here serve as vivid evidence that the fractal properties of the river network also generate the scaling properties of processes occurring in it (in this case the process of forming the water mass in the river network).

The results presented in this article indicate the proposed computer model of a fractal river network with fractal individual streams, is qualitatively adequate to describe natural river networks. We also believe that our interpretation of Hack's empirical law [8] is better substantiated than [1–3] since it takes into account the fractal properties of both the individual streams and river networks. An important result is also the identification of scaling properties of the processes whose geometrical carrier is the river network. In their aggregate the obtained results distinguish the proposed model ($\nu_\parallel = 0.74$, $\nu_\perp = 0.43$) from two classes of ramified fractal objects described by Meakin [10] which include (i) the off-lattice ballistic, ballistic lattice and Eden models ($\nu_\parallel = 0.60$, $\nu_\perp = 0.40$) and (ii) Scheidegger's river patterns ($\nu_\parallel = \frac{2}{3}$, $\nu_\perp = \frac{1}{3}$). One more distinction of our model is the existence of both self-similarity and self-affinity regions.

It should be mentioned that the scaling exponents for natural river networks ($\nu_\parallel = 0.61$, $\nu_\perp = 0.44$ for the Alto Liri basin) are rather close to those of the class (i) above, despite there being no direct analogy between river network formation and ballistic deposition. To find out if natural river networks fall into the class of universality (i) or into the new class found by us, additional data on natural river networks are necessary because data on one basin may be unrepresentative.

The authors are grateful to P G Gania and J A Antonova for their assistance in selecting and preparing the manuscript of the article.

References

[1] Kondoh H and Matsushita M 1986 *J. Phys. Soc. Japan* **55** 3289
[2] Kondoh H, Matsushita M and Fukuda Y 1987 *J. Phys. Soc. Japan* **56** 1913
[3] Meakin P, Feder J and Jossang T 1991 *Physica* **176A** 409
[4] Huber G 1991 *Physica* **170A** 463
[5] Rosso R, Bacchi B and La Barbera P 1991 *Water Resources Res.* **27** 381
[6] Feder J 1988 *Fractals* (New York: Plenum)
[7] Mandelbrot B B 1986 Self-affine fractal sets. *Fractals in Physics* (Amsterdam: North-Holland) p 3
[8] Hack J T 1957 *Geol. Surv. Prof.* Pap. 294–B
[9] Rzhanicyn N A 1960 Morphological and hydrological laws of river network structure (in Russian) (Leningrad: Gidrometeoizdatd)
[10] Meakin P 1987 *J. Phys. A: Math. Gen.* **20** L1113